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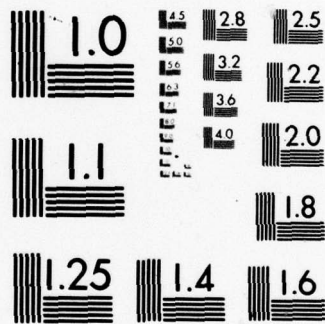
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**AN INVESTIGATION INTO THE COMBINED
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AN INVESTIGATION INTO THE COMBINED ISOTROPIC-KINEMATIC WORKHARDENING RULE

David H. Allen*

ABSTRACT

✗ The Ziegler modification of the Prager kinematic workhardening rule has been shown to be a correct statement of consistency during plastic loading. Although the same modification has been used in conjunction with a combined isotropic-kinematic workhardening rule, it has not been established that this latter model renders an acceptable statement of consistency. This paper shows that the Ziegler modification used with the combined isotropic-kinematic workhardening rule not only satisfies the consistency condition, but is also computationally more efficient than a direct employment of the consistency condition.

INTRODUCTION

↖ It has been generally accepted that the Ziegler modification¹ of the Prager kinematic workhardening rule² for rate independent plasticity satisfies the consistency condition for the case of kinematic hardening, as evidenced by the work of Hunsaker³, Vaughan⁴, Lee⁵ and others. However, it is by no means apparent that the Ziegler modification can consistently be applied to combined isotropic-kinematic hardening in the manner employed by Tanaka⁶ and later Hunsaker⁷.

In this paper, the derivation obtained by Tanaka for combined isotropic-kinematic hardening using Ziegler's modification will be reviewed. Next,

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the derivation for combined isotropic-kinematic hardening will be performed using the consistency condition. It will be shown that the derivation using Ziegler's modification is identical to the derivation using the consistency condition, and thus, the form proposed by Tanaka is a form consistent with the yield function. Finally, it will also be shown that the form derived by Tanaka is both simpler and computationally easier to employ than the formulation derived from the consistency condition.

REVIEW OF ZIEGLER'S MODIFICATION APPLIED TO COMBINED ISOTROPIC-KINEMATIC HARDENING

Tanaka has used Ziegler's modification in conjunction with the combined isotropic-kinematic hardening rule to formulate an elastic-plastic constitutive relation. That derivation will first be reviewed briefly.

Recall that for the case of combined isotropic-kinematic hardening without temperature or creep effects the yield function takes the form

$$F(S_{ij} - \alpha_{ij}) = k^2 (\int dE^P_{ij}) \quad (1)$$

where S_{ij} is the stress tensor, dE^P_{ij} is the plastic strain increment, and α_{ij} is a tensor representing the location of the center of the yield surface with respect to the origin of stress space. In addition, $k^2 (\int dE^P_{ij})$ is understood to be the size of the yield surface as a function of the history of plastic strain.

Ziegler's modification can be stated

$$\frac{\partial F}{\partial S_{ij}} dS_{ij} = c \frac{\partial F}{\partial S_{ij}} dE^P_{ij} \quad (2)$$

where the above statement can be regarded as a statement of consistency for kinematic hardening, as shown by Ziegler. However, it has not been previously shown to be a statement of consistency for combined isotropic-kinematic hardening.

Assuming small strains, the stress increment may be written as

$$dS_{ij} = D_{ijmn} (dE_{mn} - dE_{mn}^P) \quad (3)$$

where D_{ijmn} is the elastic constitutive tensor, dE_{mn} is the increment of the total strain, and dE_{mn}^P is the increment of plastic strain.

The normality condition requires that

$$dE_{ij}^P = d\lambda \frac{\partial F}{\partial S_{ij}} \quad (4)$$

where $d\lambda$ is a scalar to be determined. Now substitute equation (4) into equations (2) and (3) to obtain

$$\frac{\partial F}{\partial S_{ij}} dS_{ij} = c \frac{\partial F}{\partial S_{ij}} d\lambda \frac{\partial F}{\partial S_{ij}} \quad (5)$$

and

$$dS_{ij} = D_{ijmn} \left(dE_{mn} - d\lambda \frac{\partial F}{\partial S_{mn}} \right) \quad (6)$$

Next, substitute equation (6) into equation (5) to obtain

$$\frac{\partial F}{\partial S_{ij}} D_{ijmn} \left(dE_{mn} - d\lambda \frac{\partial F}{\partial S_{mn}} \right) = c \frac{\partial F}{\partial S_{ij}} d\lambda \frac{\partial F}{\partial S_{ij}} \quad (7)$$

Solving for $d\lambda$ yields

$$d\lambda = \frac{\frac{\partial F}{\partial S_{ij}} D_{ijmn} dE_{mn}}{\left(D_{pqrs} \frac{\partial F}{\partial S_{rs}} + c \frac{\partial F}{\partial S_{pq}} \right) \frac{\partial F}{\partial S_{pq}}} \quad (8)$$

Now substitute equation (8) into equation (6):

$$dS_{ij} = D_{ijmn} \left[dE_{mn} - \left(\frac{\frac{\partial F}{\partial S_{tu}} D_{tuvw} dE_{vw}}{D_{pqrs} \frac{\partial F}{\partial S_{rs}} + c \frac{\partial F}{\partial S_{pq}}} + c \frac{\partial F}{\partial S_{pq}} \right) \frac{\partial F}{\partial S_{mn}} \right] \quad (9)$$

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$$dS_{ij} = \left(D_{ijmn} - \frac{D_{ijvw} \frac{\partial F}{\partial S_{vw}} \frac{\partial F}{\partial S_{tu}} D_{tumn}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + c \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \right) dE_{mn} \quad (10)$$

Since c is invariant with coordinate transformation, it can be derived from equation (2) using a uniaxial test. Thus

$$c = \frac{2}{3} \frac{d\sigma_x}{d\epsilon_x^p} = \frac{2}{3} H' \quad (11)$$

where σ_x and ϵ_x^p are uniaxial stress-plastic strain data and, by definition,

$$H' = \frac{d\sigma_x}{d\epsilon_x^p} = \frac{EE^T}{E-E^T} \quad (12)$$

where E is the elastic modulus and E^T is the tangent modulus of the uniaxial stress strain curve at a given strain. Substituting equation (11) into equation (10) gives

$$dS_{ij} = \left(D_{ijmn} - \frac{D_{ijvw} \frac{\partial F}{\partial S_{vw}} \frac{\partial F}{\partial S_{tu}} D_{tumn}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + \frac{2}{3} H' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \right) dE_{mn} \quad (13)$$

This corresponds to the constitutive relation derived by Tanaka for combined isotropic-kinematic hardening.

DERIVATION OF COMBINED ISOTROPIC-KINEMATIC HARDENING RULE USING THE CONSISTENCY CONDITION

A statement of consistency during plastic loading may be obtained by differentiating equation (1):

$$\frac{\partial F}{\partial (S_{ij} - \alpha_{ij})} d(S_{ij} - \alpha_{ij}) = 2kdk \quad (14)$$

or, since the location of the center of the yield surface in stress space does not change during a load increment

$$\frac{\partial F}{\partial S_{ij}} dS_{ij} - \frac{\partial F}{\partial S_{ij}} d\alpha_{ij} = 2kdk \quad (15)$$

Recall that for kinematic hardening the yield surface does not expand so that for this special case

$$\frac{\partial F}{\partial S_{ij}} dS_{ij} - \frac{\partial F}{\partial S_{ij}} d\alpha_{ij} = 0 \quad (16)$$

Ziegler used this condition in conjunction with the assumption that

$$d\alpha_{ij} = c dE_{ij}^p \quad (17)$$

Therefore, for kinematic hardening, the consistency condition [equation (16)] becomes

$$\frac{\partial F}{\partial S_{ij}} dS_{ij} - \frac{\partial F}{\partial S_{ij}} c dE_{ij}^p = 0 \quad (18)$$

Note that equation (18) is identical to equation (2).

Equation (18) is regarded as a correct statement of consistency for kinematic hardening. Therefore, to obtain a statement of consistency for combined isotropic-kinematic hardening, substitute equation (17) into equation (15) to obtain

$$\frac{\partial F}{\partial S_{ij}} dS_{ij} - \frac{\partial F}{\partial S_{ij}} c' dE_{ij}^p = 2kdk \quad (19)$$

It should be noted that c' defined in equation (19) differs from the value obtained in Tanaka's derivation. [equation (11)].

To obtain the counterpart of equation (10) first assume as before that the stress increment is given by equation (3) and that the normality condition [equation (4)] holds. Substitute equation (4) into equation (19):

$$\frac{\partial F}{\partial S_{ij}} dS_{ij} - \frac{\partial F}{\partial S_{ij}} c' d\lambda \frac{\partial F}{\partial S_{ij}} = 2kdk \quad (20)$$

and substitute equation (6) into equation (20):

$$\frac{\partial F}{\partial S_{ij}} D_{ijmn} \left(dE_{mn} - d\lambda \frac{\partial F}{\partial S_{mn}} \right) - \frac{\partial F}{\partial S_{ij}} c' d\lambda \frac{\partial F}{\partial S_{ij}} = 2kdk \quad (21)$$

Solving for $d\lambda$ gives

$$d\lambda = \frac{\frac{\partial F}{\partial S_{ij}} D_{ijmn} dE_{mn} - 2kdk}{\left(D_{pqrs} \frac{\partial F}{\partial S_{rs}} + c' \frac{\partial F}{\partial S_{pq}} \right) \frac{\partial F}{\partial S_{pq}}} \quad (22)$$

Next, apply equation (22) to equation (6) to obtain

$$dS_{ij} = D_{ijmn} \left[dE_{mn} - \left(\frac{\frac{\partial F}{\partial S_{tu}} D_{tuvw} dE_{vw} - 2kdk}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + c' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \right) \frac{\partial F}{\partial S_{mn}} \right] \quad (23)$$

Thus,

$$dS_{ij} = \left(D_{ijmn} - \frac{D_{ijvw} \frac{\partial F}{\partial S_{vw}} \frac{\partial F}{\partial S_{tu}} D_{tumn}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + c' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \right) dE_{mn} + \frac{2kdk D_{ijmn} \frac{\partial F}{\partial S_{mn}}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + c' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \quad (24)$$

Although the above equation appears to contradict equation (10), it should be reiterated that the c' used in this derivation differs from that used to obtain equation (10). Now solve equation (19) for c' :

$$c' = \frac{\frac{\partial S_{ij}}{\partial S_{ij}} \frac{\partial F}{\partial S_{ij}}}{dE_{pq}^p \frac{\partial F}{\partial S_{pq}}} - \frac{2kdk}{dE_{ij}^p \frac{\partial F}{\partial S_{ij}}} \quad (25)$$

For a uniaxial tensile test, it can be shown that

$$c' = \frac{2}{3} H' - \frac{2kdk}{dE_{ij}^p \frac{\partial F}{\partial S_{ij}}} \quad (26)$$

where H' is as defined previously. Substituting equation (4) into equation (25) gives

$$c' = \frac{2}{3} H' - \frac{2kdk}{d\lambda \frac{\partial F}{\partial S_{ij}} \frac{\partial F}{\partial S_{ij}}} \quad (27)$$

Now substitute equation (27) into equation (24). The result is

$$dS_{ij} = \left(D_{ijmn} - \frac{D_{ijvw} \frac{\partial F}{\partial S_{vw}} \frac{\partial F}{\partial S_{tu}} D_{tumn}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + \frac{2}{3} H' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}} - \frac{2kdk}{d\lambda}} \right) dE_{mn} \\ + \frac{2kdk D_{ijmn} \frac{\partial F}{\partial S_{mn}}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + \frac{2}{3} H' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}} - \frac{2kdk}{d\lambda}} \quad (28)$$

The above constitutive relation applies for combined isotropic-kinematic hardening and satisfies the consistency condition.

COMPARISON OF THE TWO CONSTITUTIVE RELATIONS

A comparison of equations (28) and (13) would tend to lead one to believe that equation (13) does not satisfy the consistency condition because the term containing $2kdk$ is conspicuously absent and in order for equation (13) to be consistent it must be identical to equation (28). A closer inspection reveals that the denominators of the two equations differ. Therefore, as a first step, rewrite equation (28) so that the denominator is of a form identical to that in equation (13). To do this first subtract $D_{ijmn} dE_{mn}$ from both sides and multiply both sides of equation (28) by the denominator. The resulting equation is

$$\begin{aligned}
& (dS_{ij} - D_{ijmn} dE_{mn}) \left(\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + \frac{2}{3} H' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}} \right) = \\
& - \left(D_{ijvw} \frac{\partial F}{\partial S_{vw}} \frac{\partial F}{\partial S_{tu}} D_{tumn} \right) dE_{mn} + 2kdk D_{ijmn} \frac{\partial F}{\partial S_{mn}} \\
& + \frac{2kdk}{d\lambda} (dS_{ij} - D_{ijmn} dE_{mn})
\end{aligned} \tag{29}$$

Now divide both sides by the second term in parentheses:

$$\begin{aligned}
dS_{ij} - D_{ijmn} dE_{mn} = & - \frac{D_{ijvw} \frac{\partial F}{\partial S_{vw}} \frac{\partial F}{\partial S_{tu}} D_{tumn} dE_{mn}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + \frac{2}{3} H' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \\
& + 2kdk \left[\frac{\left(D_{ijmn} \frac{\partial F}{\partial S_{mn}} + \frac{dS_{ij}}{d\lambda} - \frac{D_{ijmn} dE_{mn}}{d\lambda} \right)}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + \frac{2}{3} H' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \right]
\end{aligned} \tag{30}$$

Add $D_{ijmn} dE_{mn}$ to both sides and substitute the normality condition into the numerator of the last term:

$$\begin{aligned}
dS_{ij} = & \left(D_{ijmn} - \frac{D_{ijvw} \frac{\partial F}{\partial S_{vw}} \frac{\partial F}{\partial S_{tu}} D_{tumn}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + \frac{2}{3} H' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \right) dE_{mn} \\
& + 2kdk \left[\frac{\frac{D_{ijmn} dE_{ij}^P}{d\lambda} + \frac{D_{ijmn}}{d\lambda} (dE_{ij} - dE_{ij}^P) - \frac{D_{ijmn} dE_{mn}}{d\lambda}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + \frac{2}{3} H' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \right]
\end{aligned} \tag{31}$$

It can be seen that the numerator of the second term on the right hand side is identically zero. Thus, equation (31) reduces to

$$dS_{ij} = \left(D_{ijmn} - \frac{D_{ijvw} \frac{\partial F}{\partial S_{vw}} \frac{\partial F}{\partial S_{tu}} D_{tumn}}{\frac{\partial F}{\partial S_{pq}} D_{pqrs} \frac{\partial F}{\partial S_{rs}} + \frac{2}{3} H' \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \right) dE_{mn} \tag{32}$$

and equation (32) is now identical to equation (13).

CONCLUSION

It has been shown that the combined isotropic-kinematic hardening rule used in conjunction with Ziegler's modification yields a constitutive relation which is identical to that obtained using the consistency condition. Therefore, it can be said that the formulation utilizing Ziegler's modification also satisfies the consistency condition. The two methods are identical because Ziegler's modification [equation (2)] can be used as a statement of consistency whenever the terms in the consistency condition are a function of the plastic strain increment. Since the yield surface size, i.e., the term containing k in equation (15), is a function of the plastic strain increment, it can be correctly accounted for by the fact that the value of c is altered in Ziegler's modification.

For cases where the consistency condition contains parameters which are not a function of the plastic strain increment (e.g., yield surface temperature dependence) it would not be proper to use Ziegler's statement without still further modification.

The question remains as to which of the two formulations presented herein is more desirable to use. This question is disposed of by noting first that the formulation presented by Hunsaker entails a c' value which is computationally simpler than the c value obtained in the second formulation. Second, it is apparent that in the second formulation [equation (28)], the terms containing dk can only be arrived at through an iterative technique. This problem is not encountered in the Hunsaker formulation because the stress increment is not a function of the increment in yield surface size. Thus, it is apparent that the formulation presented by Hunsaker is not only consistent, but also computationally concise.

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